Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-11-01

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

(a) We can't use FTLI because $\langle 0, x-2\rangle$ is not a conservative vector field, and we can't use Green's Theorem because the curve C is not a loop. Sure, we could close the loop in theory, say by including the line segments from $(3,0)$ to $(0,0)$ and then to $(0,3)$ (see part (f) for instance), but that's an awful lot of extra work in this case and the direct parametrization $x = 3\cos t, y = 3\sin t, 0 \le t \le \pi/2$ is not hard to use:

$$\int_C \langle 0, x-2\rangle \cdot \mathrm{d}\mathbf{r} = \int_0^{\pi/2} (3\cos t - 2)(3\cos t)\,\mathrm{d}t.$$

(b) Let $f(x,y) = \frac{1}{3}x^3 + \frac{1}{4}y^4$. Then $\nabla f = \langle x^2, y^3\rangle$ so the problem is easily solved by FTLI:

$$\int_C \langle x^2, y^3\rangle \cdot \mathrm{d}\mathbf{r} = f(0,3) - f(3,0).$$

(c) As this is an integral with respect to ds, the fancy tools of FTLI and Green's theorem are not available to us. So there is no choice but to directly parametrize:

$$\int_D x\,\mathrm{d}s = \int_{-1}^{2} t\sqrt{2}\,\mathrm{d}t + \int_{-1}^{2} t\sqrt{4t^2+1}\,\mathrm{d}t.$$

Here I've used $x = t, y = t+2, -1 \le t \le 2$ to parametrize the line, and $x = t, y = t^2, -1 \le t \le 2$ to parametrize the parabola. Notice that for convenience I've parametrized the parabola left to right instead of the way it is actually traced (right to left), which yields the same answer because this is an integral with respect to ds.

(d) This is a work integral over a closed loop, so we are poised to use Green's Theorem.

$$\int_D \langle \sin(x^3), xy^2\rangle \cdot \mathrm{d}\mathbf{r} = \int_{-1}^{2}\int_{x^2}^{x+2} y^2\,\mathrm{d}y\,\mathrm{d}x.$$

(e) We already saw in (b) that this vector field is conservative. So just use FTLI as before.

(f) Let R be the filled-in trapezoid with the four points in the problem as corners. The positively oriented boundary ∂R is then the union of $-E$ (the reverse of E) with the line L from $(-2,0)$ to $(2,0)$. So Green's Theorem says

$$\iint_R y^2\,\mathrm{d}x\,\mathrm{d}y = \int_L \langle \sin(x^3), xy^2\rangle \cdot \mathrm{d}\mathbf{r} - \int_E \langle \sin(x^3), xy^2\rangle \cdot \mathrm{d}\mathbf{r}.$$

The integral over L is actually equal to zero in this case (why?), hence

$$\int_E \langle \sin(x^3), xy^2\rangle \cdot \mathrm{d}\mathbf{r} = -\iint_R y^2\,\mathrm{d}x\,\mathrm{d}y.$$

Answers to computations

Problem 1.

(a) Let R denote the region *enclosed* by the polygon, so that ∂R is the polygon, oriented counterclockwise. Green's Theorem tells us that the area of R can be computed as

$$\iint_R 1\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} \langle -y/2, x/2\rangle \cdot \mathrm{d}\mathbf{r}.$$

The RHS is a sum of n line integrals, one for each edge of the polygon. For example, the edge from (x_1, y_1) to (x_2, y_2) can be parametrized as

$$\langle x, y\rangle = \langle x_1, y_1\rangle + t\langle x_2 - x_1, y_2 - y_1\rangle,\ 0 \le t \le 1$$

(this should look familiar from Chapter 12). Hence the line integral over this edge can be rewritten as

$$\int_0^1 \frac{1}{2}\langle -y_1 - ty_2 + ty_2, x_1 + tx_2 - tx_1\rangle \cdot \langle x_2 - x_1, y_2 - y_1\rangle\,\mathrm{d}t = \frac{1}{2}\int_0^1 (x_1y_2 - x_2y_1)\,\mathrm{d}t = \frac{1}{2}(x_1y_2 - x_2y_1).$$

Going over all of the edges,

$$\int_{\partial R} \langle -y/2, x/2 \rangle \cdot d\mathbf{r} = \frac{1}{2}(x_1 y_2 - x_2 y_1) + \frac{1}{2}(x_2 y_3 - x_3 y_2) + \cdots + \frac{1}{2}(x_{n-1} y_n - x_n y_{n-1}) + \frac{1}{2}(x_n y_1 - x_1 y_n).$$

(b) This is because $\langle x_i, y_i, 0 \rangle \times \langle x_j, y_j, 0 \rangle = \langle 0, 0, x_i y_j - x_j y_i \rangle$. I drew a picture explaining the whole expression in class: essentially this computes the area of the polygon via a *triangulation* (it is a bit more subtle if the polygon does not enclose the origin, but attention to signs still makes things work out in the end).

(c) These just reduce to $\int_{\partial R} x \, dy$ and $-\int_{\partial R} y \, dx$; the geometric interpretations of these were covered in Chapter 10. (Can you draw a picture and explain why the second one needs the minus sign?)